

# 2-D constrained Navier-Stokes equation and intermediate asymptotics

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## Abstract

We introduce a modified version of the two-dimensional Navier-Stokes equation, preserving energy and momentum of inertia, which is motivated by the occurrence of different dissipation time scales and related to the gradient flow structure of the 2-D Navier-Stokes equation. The hope is to understand intermediate asymptotics. The analysis we present here is purely formal. A rigorous study of this equation will be done in a forthcoming paper.

## 1 Introduction

The two-dimensional incompressible Euler equation in vorticity formulation reads

$$(\partial_t + u \cdot \nabla)\omega(x, t) = 0. \quad (1.1)$$

Here  $x \in \mathbb{R}^2$ ,  $t \in \mathbb{R}^+$  and  $u = u(x, t) \in \mathbb{R}^2$  is the velocity field defined as:

$$u = \nabla^\perp \psi, \quad \psi = -\Delta^{-1}\omega. \quad (1.2)$$

Explicitely, we have :

$$u = K * \omega, \quad K(x) = -\frac{1}{2\pi} \nabla^\perp \log |x| = -\frac{1}{2\pi} \frac{x^\perp}{|x|^2}.$$

This equation is formally hamiltonian (we refer for example to [MTWR]). The Hamiltonian is the energy

$$E(\omega) = \frac{1}{2} \int \psi \omega dx. \quad (1.3)$$

By Noether Theorem, the center of mass  $M = \int x \omega$  (which is related to the invariance of  $E$  with respect to the group of translations) and the momentum of inertia (which is related to the invariance of  $E$  with respect to the group of rotations)  $I(\omega) = \frac{1}{2} \int |x - M|^2 \omega dx$  are also conserved. Moreover,

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due to the degeneracy of the Poisson bracket, one of the main feature of (1.1) is the presence of an infinite number of conserved quantities, sometimes called Casimir: all the integrals of the form

$$F_\phi(\omega) = \int \phi(\omega) dx$$

are conserved. For this reason, the rigorous study of the large time behaviour of the solution of (1.1) and hence the justification from (1.1) of the presence of coherent structures observed in real and numerical experiments (see e.g. [MS]) remains widely open. In the following, we shall focus on non-negative solutions, we shall assume that  $\omega$  is a probability distribution so that  $\omega$  is non negative and  $\int \omega = 1$  constantly in time. We also fix the reference frame in such a way that  $\int x\omega = 0$  for every time.

One attempt to justify the appearance of these coherent structure is due to Onsager [O], see also [LP], [MJ]. The main idea is to replace the Euler equation (1.1) by the system of  $N$  point vortices and to study the Statistical Mechanics of these point vortices. In a limit  $N \rightarrow +\infty$ , the Gibbs measure associated to the point vortices concentrates to some special stationary solutions of the Euler equation (called mean field solutions), this was rigorously justified in [CLMP1], [CLMP2], [K] and [KJ]. These states are under the form:

$$\omega = \frac{e^{b\psi + a\frac{|x|^2}{2}}}{Z} \quad (1.4)$$

where

$$Z = \int e^{b\psi + a\frac{|x|^2}{2}}$$

is a normalization to have  $\int \omega = 1$ . Recalling that  $\omega = -\Delta\psi$ , we realize that eq.n (1.4) is a nonlinear elliptic equation. The study of this equation in connection with the variational principles arising from statistical mechanics was performed in [CLMP1], [CLMP2]. Let us summarize the main results. We define the free-energy functional as

$$F_{(b,a)}(\omega) = S(\omega) - bE(\omega) - aI(\omega) \quad (1.5)$$

for a given pair  $a < 0$  and  $b > 0$ .  $F_{(b,a)}$  is defined on the space  $\Gamma$  of all the probability densities on  $\mathbb{R}^2$  with finite entropy, energy and moment of inertia. We define the canonical variational principle as

$$F(b, a) = \inf_{\omega \in \Gamma} F_{(b,a)}(\omega). \quad (1.6)$$

Next, for  $E \in \mathbb{R}$  and  $I > 0$  let us introduce the set

$$\Gamma_{(E,I)} = \{\omega \in \Gamma | E(\omega) = E, I(\omega) = I\} \quad (1.7)$$

and consider the microcanonical variational principle

$$S(E, I) = \inf_{\omega \in \Gamma_{(E,I)}} S(\omega). \quad (1.8)$$

The above variational problems can be related to the solutions of the Mean-Field equation (1.4). Moreover, in the whole space  $\mathbb{R}^2$ , we have also the equivalence of the ensembles:

**Theorem 1 (CLMP1, CLMP2)** *For  $a < 0$  and  $0 < b < 8\pi$ :*

*i) There exists a unique, radially symmetric minimizer  $\omega = \omega_{(b,a)} \in \Gamma$  of the problem (1.6) which is the unique radially symmetric solution to eq.n (1.4).*

*ii) When  $b \rightarrow 8\pi$ ,  $\omega$  converges (weakly) to a  $\delta$  at the origin.*

*iii)  $F(b, a)$  is a concave smooth function and*

$$\frac{\partial F}{\partial a} = I(\omega_{(b,a)}), \quad \frac{\partial F}{\partial b} = -E(\omega_{(b,a)})$$

*iv) For  $E \in \mathbb{R}$  and  $I > 0$  define*

$$S^*(I, E) = \sup_{a,b} (F(b, a) + bE + aI)$$

*and denote by  $b(I, E)$  and  $a(I, E)$  the unique maximizers. Then  $S(I, E) = S^*(I, E)$  and hence  $S$  is a smooth convex function.*

*v) The variational problem (1.8) has a unique minimizer  $\omega(I, E)$  and*

$$\omega(I, E) = \omega_{(b(I,E), a(I,E))}.$$

Note that when  $b \leq 0$  the theory is easier. Indeed the functional  $F_{(b,a)}(\omega)$  is convex so the minimization problem is standard and eq.n (1.4) has a unique (radial) solution [GL]. We also point out that the equivalence between (1.6) and (1.8) which is established in v) is also useful to establish the existence of a minimizer for (1.8). Indeed, in an unbounded domain, the existence of a minimizer for (1.8) seems difficult to establish directly because of the absence of higher moment control which would allow to pass to the limit in the constraint for the moment of inertia. Note also that eq.n (1.4) has a natural statistical mechanical interpretation, its solutions being Gibbs states with a self-consistent interaction. Therefore  $-b$  is an inverse temperature. Hence  $b > 0$  implies negative temperature states, as predicted by Onsager [O] in terms of point vortex theory.

A rigorous justification of the fact that the solutions of the mean field equation plays a special part in the large time behaviour of the Euler equation still seems an out of reach problem. An attempt towards the justification of the fact that the states (1.4) play a special part in the 2D turbulence could come from the study of the intermediate behaviour of the Navier-Stokes equation. Indeed, for the Navier-Stokes equation,

$$(\partial_t + u \cdot \nabla)\omega(x, t) = \nu \Delta \omega(x, t), \quad (1.9)$$

$\int \omega$  and  $\int x\omega$  are still conserved so that we can still consider non-negative solutions so that  $\int \omega = 1$  and  $\int x\omega = 0$ . Nevertheless, due to the dissipation term in the right hand side of eq.n (1.9), the asymptotic behavior of the solutions is trivial, namely  $\omega(x, t) \rightarrow 0$  pointwise and in the  $L^p$  sense for

$p > 1$ . Consequently, one can hope to observe the mean field solutions only as intermediate states. To formalize this idea, let us notice that the momentum of inertia  $I$  increases by a constant rate:

$$\dot{I}(\omega) = 2\nu \quad (1.10)$$

consequently it can be considered as constant for times  $\ll 1/\nu$ . In a similar way,  $E$  and  $S$  are dissipated with rates

$$\dot{E} = -\nu \int \omega^2, \quad \dot{S} = -\nu \int \frac{|\nabla \omega|^2}{\omega}. \quad (1.11)$$

Looking at eq.ns (1.11), one realizes that the energy could also evolve on a different and longer scale of times with respect to  $S$  (whenever the last term in (1.11) dominates on the first one). This would suggest to consider, in the first approximation,  $E$  and  $I$  as constant, by looking at a master equation which modifies the Navier-Stokes equation leaving constant both energy and moment of inertia, but retaining all the other features of the Navier-Stokes dynamics. The derivation of such an equation, based on geometric arguments is the aim of the following section.

## 2 Derivation of the model

A natural and fruitful way to approach the problem is to invoke a recent characterization of the Navier-Stokes equation connected with the mass transport problem and the associated differential calculus introduced in [Ot]. We follow the excellent monographies [V], [AGS] for outlining the main ideas.

Let  $\mathcal{M}$  be the manifold of the probability measures on  $\mathbb{R}^2$ . One can formally give to  $\mathcal{M}$  a structure of Riemannian manifold. For any  $\rho \in \mathcal{M}$  we parametrize the tangent space as

$$T_\rho \mathcal{M} = \{\dot{\rho} | \dot{\rho} = -\operatorname{div} u\}. \quad (2.1)$$

Eq.n (2.1) expresses the tangent space to any point  $\rho$  of  $\mathcal{M}$  as mass preserving velocity vectors  $\dot{\rho}$ . Next, we define a Riemannian metric. On the tangent space  $T_\rho \mathcal{M}$ , we define a scalar product by

$$\langle \dot{\rho}_1, \dot{\rho}_2 \rangle_W = \int \rho^{-1} u_1 \cdot u_2 \, dx, \quad (2.2)$$

being  $\dot{\rho}_i = -\operatorname{div} u_i$ ,  $i = 1, 2$ . The gradient  $\nabla_W$  with respect to this Riemannian metric of a functional  $F : \mathcal{M} \rightarrow \mathbb{R}$ , is defined as:

$$\langle \nabla_W F, \dot{\rho} \rangle_W = DF \cdot \dot{\rho} = - \int \frac{\delta F}{\delta \rho} \operatorname{div} u,$$

where  $DF$  is the differential of the map  $F$  and  $\delta F / \delta \rho$  the usual variational derivative. An explicit computation shows that

$$\nabla_W F = -\operatorname{div} \left[ \rho \nabla \frac{\delta F}{\delta \rho} \right]. \quad (2.3)$$

In a similar way, we can define on  $T_\rho \mathcal{M}$  a skew-symmetric operator  $J_W$  by

$$J_W \dot{\rho} = -\operatorname{div} \left( u^\perp \right), \quad \dot{\rho} = -\operatorname{div} u \quad (2.4)$$

where  $u^\perp = (u_2, -u_1)$ . Note that this yields in particular the expression

$$J_W \nabla_W F = -\operatorname{div} \left[ \rho \nabla^\perp \frac{\delta F}{\delta \rho} \right]. \quad (2.5)$$

Gradient flows with respect to a functional  $F$  are the solutions to

$$\partial_t \rho = -\nabla_W F. \quad (2.6)$$

and are dissipative in the sense that

$$\frac{d}{dt} F = - \int \omega \left| \nabla \frac{\delta F}{\delta \rho} \right|^2 \leq 0$$

whereas hamiltonian flows defined by

$$\partial_t \rho = J_W \nabla_W F$$

are conservative since

$$\frac{d}{dt} F = 0.$$

Since  $\delta E / \delta \omega = \psi$ , we get from (2.5) that

$$\operatorname{div} \left[ \omega \nabla^\perp \psi \right] = J_W \nabla_W E$$

and hence, the Euler equation can be interpreted as an Hamiltonian flow in this framework.

Next, to interpret the dissipative part of the Navier-Stokes equation, we notice that

$$\Delta \omega = \operatorname{div} \left( \omega \nabla \frac{\delta S}{\delta \omega} \right) = -\nabla_W S$$

where  $S$  is the entropy functional.

In conclusion the Navier-Stokes equation can be expressed in terms of a gradient and an anti-gradient flow :

$$\partial_t \omega = -\nu \nabla_W S + J_W \nabla_W E. \quad (2.7)$$

According to the previous discussion we assume that the energy and the moment of inertia are varying much more slowly than the entropy functional. Therefore it may be useful to derive an effective equation according to the following prescription. We shall take the orthogonal projection (with respect to the scalar product (2.2)) of the vector field in the right hand side of eq.n (2.7) on the manifold  $E = \text{const}$  and  $I = \text{const}$ , with the aim to characterize the states which are close to the true dynamics for a large interval of times (coherent structures), as asymptotic states of the

new dynamics. We shall see that such states are the solutions to the mean-field equation (1.4). Since:

$$\begin{aligned}\nabla_W E(\omega) &= -\operatorname{div}\left[\omega \nabla \frac{\delta E(\omega)}{\delta \omega}\right] = -\operatorname{div}[\omega \nabla \psi], \\ \nabla_W I(\omega) &= -\operatorname{div}\left[\omega \nabla \frac{\delta I(\omega)}{\delta \omega}\right] = -\operatorname{div}\left[\omega \nabla \frac{x^2}{2}\right] = -\operatorname{div}[\omega x].\end{aligned}$$

we find that

$$\langle \nabla_W E, J_W \nabla_W E \rangle_W = 0 \quad \langle \nabla_W I, J_W \nabla_W E \rangle_W = 0,$$

and hence  $J_W \nabla_W E$  is tangent to the manifold  $E = \text{const}$ ,  $I = \text{const}$ . On the other hand the projection of  $\nabla_W S$  on the tangent space of such a manifold is of the form

$$\nabla_W S - b \operatorname{div}(\omega \nabla \psi) - a \operatorname{div}(\omega x),$$

with  $a$  and  $b$  two suitable multipliers. As a consequence, the equation we are looking for is:

$$\begin{aligned}\partial_t \omega + u \cdot \nabla \omega &= \nu \operatorname{div}(\nabla \omega - b \omega \nabla \psi - a \omega x) \\ &= \nu \operatorname{div}\left[\omega \nabla \left(\log \omega - b \psi - a \frac{x^2}{2}\right)\right],\end{aligned}\tag{2.8}$$

with  $a$  and  $b$  to be determined by the simultaneous conservation of  $E$  and  $I$ . A straightforward computation yields:

$$b = \frac{2I \int \omega^2 + 2V}{2I \int \omega |\nabla \psi|^2 - V^2}, \quad a = -\frac{2 \int \omega |\nabla \psi|^2 + V \int \omega^2}{2I \int \omega |\nabla \psi|^2 - V^2},\tag{2.9}$$

where

$$V = \int \omega x \cdot \nabla \psi = \int dx \int dy \omega(x) \omega(y) x \cdot \nabla g(x - y) = -\frac{1}{4\pi}.\tag{2.10}$$

Note that we have by the Cauchy-Schwarz inequality

$$V^2 = \left(\int \omega x \cdot \nabla \psi\right)^2 \leq 2I \int \omega |\nabla \psi|^2,\tag{2.11}$$

and hence  $b$  is positive if inequality (2.11) holds strictly and  $\int \omega^2 > \frac{1}{4\pi I}$ .

We point out that eq.n (2.8) has been derived in [Ch2] (see also [Ch1]), by using different arguments than those of the present paper.

Note also that another class of similar equations was introduced in [RS] and studied mathematically in [MR]. Nevertheless, despite to some formal analogy the mathematical properties of the equations studied in [MR] are very different from the ones presented here.

Let us now discuss what we may expect about the asymptotic behavior of eq.n (2.8). The main feature of eq.n (2.8) is the decay of the entropy functional. Indeed, we have

$$\frac{dS(\omega)}{dt} = \frac{dS(\omega)}{dt} - b \frac{dE(\omega)}{dt} - a \frac{dI(\omega)}{dt} =$$

$$\begin{aligned}
& \nu \int \left( \frac{\delta S}{\delta \omega} - b \frac{\delta E}{\delta \omega} - a \frac{\delta I}{\delta \omega} \right) \operatorname{div} \left[ \omega \nabla \left( \frac{\delta S}{\delta \omega} - a \frac{\delta I}{\delta \omega} - b \frac{\delta E}{\delta \omega} \right) \right] = \\
& -\nu \int \omega \left| \nabla \left( \frac{\delta S}{\delta \omega} - a \frac{\delta I}{\delta \omega} - b \frac{\delta E}{\delta \omega} \right) \right|^2 = \\
& -\nu \int \omega \left| \nabla \left( \log \omega - b\psi - a \frac{|x|^2}{2} \right) \right|^2.
\end{aligned}$$

In particular, this suggests formally that the asymptotic states satisfy

$$\omega \nabla \left( \log \omega - b\psi - a \frac{|x|^2}{2} \right) = 0$$

and hence the mean field equation (1.4).

We remark that the procedure of constructing dissipative equations leaving invariant a given quantity is not unique. For instance the heat equation in  $\mathbb{R}^2$ :

$$\partial_t \omega = \Delta \omega$$

can be modified to leave invariant  $I$  according to the procedure suggested by the gradient flow structure. The result is

$$\partial_t \omega = \operatorname{div}(\nabla \omega - a \omega x), \quad (2.12)$$

where

$$a = -\frac{1}{I}.$$

On the other hand we could also have

$$\partial_t \omega = \partial_{\theta, \theta}^2 \omega. \quad (2.13)$$

Note that eq.ns (2.12) and (2.13) have different asymptotic states.

An attempt to characterize an intermediate asymptotics was presented by Gallay and Wayne [GW] according to the following ideas.

It is well known that a special solution to eq.n (1.9) (for  $\nu = 1$ ) is given by the so called Oseen vortex:

$$\omega(x, t) = \frac{1}{4\pi(t+1)} e^{-\frac{|x|^2}{4(t+1)}}.$$

Note that this is also a solution to the heat equation. It was shown in [GW] that this solution describes the long time asymptotic of the Navier-Stokes equation in  $L^1$ . Indeed, with the change of variables

$$\xi = \frac{x}{\sqrt{1+t}}; \quad \tau = \log(1+t), \quad \omega(x, t) = (1+t)^{-1} w(\xi, \tau),$$

the Navier-Stokes equation in the new variables is under the form :

$$\partial_\tau w + v \cdot \nabla_\xi w = \Delta_\xi w + \nabla_\xi \cdot \left( \frac{1}{2} \xi w \right). \quad (2.14)$$

It is possible to show that  $w \rightarrow W$  in  $L^1$  as  $t \rightarrow \infty$ , where  $W(\xi)$  is the rescaled Oseen vortex. As a consequence the Oseen vortex can be thought as characterizing an intermediate asymptotics before the dissipation scale. Note that  $W$  is also a solution to (1.4) for  $b = 0$ .

This analysis enters perfectly in the context of the projected gradient flows. Indeed neglecting the energy, the mere constance of  $I$  yields

$$\partial_t \omega + u \cdot \nabla \omega = \Delta \omega + \frac{1}{I} \nabla \cdot (\omega x), \quad (2.15)$$

that is eq.n (2.14) for  $I = 2$ . In this particular case we have seen how eq.n (2.14) can be obtained also by a simple change of variables, due to the fact that the dissipation rate of  $I$  is constant. Roughly speaking the analysis we present here is an attempt to see what happens before the appearance of the Oseen vortex by projecting on a less robust manifold. Indeed one could argue that  $I$  is more stable than  $E$  in many interesting physical situations. If so eq.n (2.8) should be more appropriate on the time scale when  $E$  is practically constant, while eq.n (2.14) should describe the fluid when  $E$  start to be dissipated at constant  $I$ . After that, everything disappears.

It would be interesting to look for a numerical evidence of this fact, if true.

The formal derivation of eq.n (2.8) is also in agreement with the stochastic vortex theory as we are going to illustrate.

As explained previously, the procedure of constructing dissipative equation leaving invariant a given quantity is not unique. Equation (2.8) has been interpreted as a constrained Navier-Stokes flow and it turns out that the asymptotic states are solution of the microcanonical variational principle. Since this variational principle is obtained by Mean Field limit from the statistical theory of the point vortices, it is interesting to see how equation (2.8) is related to the theory of point vortices. Now we show how, at least formally, the structure of the constrained gradient flow is compatible with the discretization of the Navier-Stokes equation obtained by mean of the stochastic vortex theory. Since in this context we are not interested in the asymptotic behavior, we limit ourselves in considering the energy constraint only. Also the viscosity does not play any role so we set  $\nu = 1$ .

Consider  $N$  stochastic vortices in  $\mathbb{R}^2$ . They obey the stochastic differential eq.n:

$$dx_i = \frac{1}{N} \sum_j \nabla^\perp g(x_i - x_j) dt + \sqrt{2} dw_i \quad (2.16)$$

where  $\{w_i\}_{i=1}^N$  are  $N$  independent standard Brownian motions. Here  $g(x)$  is a regularization of the Green function  $-\frac{1}{2\pi} \log |x|$ . It is well known ([MP2], [Os]...) that the empirical random measure

$$\mu_N(dx, t) = \frac{1}{N} \sum_{j=1}^N \delta(x_j(t) - x) dx \quad (2.17)$$

approaches the solution to the (regularized) Navier-Stokes equation, if it happens at time zero.



We now consider the mean-field energy:

$$H(x_1 \dots x_N) = \frac{1}{N} \sum_{j < r} g(x_j - x_r).$$

In order to guarantee the condition

$$dH = 0$$

a short computation using the Ito formula shows that we have to modify the process according to:

$$\begin{aligned} dx_i = & \frac{1}{N} \sum_j [\nabla^\perp g(x_i - x_j) - b_N(t) \nabla g(x_i - x_j)] dt + \sqrt{2} dw_i + \\ & \sum_{i,j} D_{i,j}^2 H \frac{\nabla_i H \cdot \nabla_j H}{|\nabla H|^4} dt - \frac{1}{|\nabla H|^2} \sum_j \nabla_j H \cdot dw_j \nabla_i H, \end{aligned} \quad (2.18)$$

where

$$b_N(t) = \frac{\Delta H}{|\nabla H|^2} = \frac{\int d\mu_N \Delta g * \mu_N}{\int d\mu_N |\nabla g * \mu_N|^2}.$$

Note that the above expression makes sense only if we regularize  $g$  and this explains why we did it.

We observe that an analysis on the size of the last two terms in (2.18) shows that they should be negligible in the limit  $N \rightarrow \infty$ . Thus we introduce the essential process defined by

$$dy_i = \frac{1}{N} \sum_j [\nabla^\perp g(y_i - y_j) - b_N(t) \nabla g(y_i - y_j)] dt + \sqrt{2} dw_i.$$

Now if we replace  $\mu_N(dx, t)$  by  $\omega(x, t)dx$  in the limit  $N \rightarrow \infty$ , each process  $x_j$  approaches the nonlinear (in the McKean sense, see [Mc],[MP2],[Os]) process solution of

$$dy = [\nabla^\perp g * \omega(y) - b(t) \nabla g * \omega(y)] dt + \sqrt{2} dw. \quad (2.19)$$

where

$$b(t) = \frac{\int dx \omega \Delta g * \omega}{\int dx \omega |\nabla g * \omega|^2}.$$

From eq.n (2.19), we also derive the backward Kolmogorov equation for the probability distribution  $\omega$  of the proces  $y$ :

$$\partial_t \omega + \nabla \cdot (u\omega) = \nabla \cdot (\nabla \omega - b \omega \nabla g * \omega). \quad (2.20)$$

where  $u = \nabla^\perp g * \omega$ . It is remarkable that such an equation is the microcanonical equation (2.8), for  $a = 0$ , if we replace  $g$  by the true Green function in eq.n (2.20),

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